SOLUTION BY EDWARD H. WORTHINGTON, University of Pennsylvania.

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx.$$

Hence,

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1 \qquad \text{and} \qquad \Gamma'(n) = \int_0^\infty x^{n-1}e^{-x}\log x dx,$$

since differentiation may be carried under the sign.

Since

$$\log x = \int_0^\infty \frac{e^{-\alpha} - e^{-\alpha x}}{\alpha} d\alpha, \qquad \Gamma'(n) = \int_0^\infty x^{n-1}e^{-x}\int_0^\infty \frac{e^{-\alpha} - e^{-\alpha x}}{\alpha} d\alpha dx.$$

$$\therefore \quad \Gamma'(1) = \int_0^\infty \int_0^\infty e^{-x}\frac{e^{-\alpha} - e^{-\alpha x}}{\alpha} dx d\alpha = \Gamma(1)\int_0^\infty \left(e^{-\alpha} - \frac{1}{1+\alpha}\right)\frac{d\alpha}{\alpha}.$$

(1) Hence, $\dfrac{\Gamma'(1)}{\Gamma(1)} = \displaystyle\int_0^\infty \left(e^{-\alpha} - \frac{1}{(1+\alpha)}\right)\frac{d\alpha}{\alpha}$; also $\dfrac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = \displaystyle\int_0^\infty \left(e^{-\alpha} - \frac{1}{(1+\alpha)^{1/2}}\right)\frac{d\alpha}{\alpha}$

(2) and $\dfrac{\Gamma'(1)}{\Gamma(1)} - \dfrac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = \displaystyle\int_0^\infty \left(-\frac{1}{1+\alpha} + \frac{1}{(1+\alpha)^{1/2}}\right)\frac{d\alpha}{\alpha}.$

Changing $1 + \alpha$ to $1/\alpha$, (2) becomes, if the primes are dropped, and convergency considered,

$$\lim_{\epsilon \doteq 0}\left[\int_{1-\epsilon}^0 \frac{1-\alpha^{-1/2}}{1-\alpha} d\alpha = \int_{1-\epsilon}^0 \frac{d\alpha}{-\alpha^{1/2}(\alpha^{1/2}+1)} = \int_{1-\epsilon}^0 -\alpha^{-1/2}[1 - \alpha^{1/2} + \alpha - \alpha^{3/2} + \alpha^2 \cdots]d\alpha\right]$$
$$= +2[1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} \cdots] = +2\log_e 2.$$

Since $\log_e 2 = 0.69315$

$$\therefore \quad \frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = +1.38630.$$

Also solved by G. PAASWELL, H. M. TERRILL, and V. M. SPUNAR.

NUMBER THEORY.

254. Proposed by HORACE OLSON, Chicago, Ill.

Find three integers x, y, z, such that $x^2 + y^2$, $x^2 + z^2$, $y^2 + z^2$, and $x^2 + y^2 + z^2$ are all perfect squares.

SOLUTION BY V. M. SPUNAR, Chicago, Ill.

Let

$$x^2 + y^2 = \square = a^2, \quad (1) \qquad y^2 + z^2 = \square = c^2, \quad (3)$$
$$x^2 + z^2 = \square = b^2, \quad (2) \qquad x^2 + y^2 + z^2 = \square = d^2. \quad (4)$$

A complete solution of (4) is as follows:

$$x = m^2 + n^2 - p^2 - q^2, \quad y = 2(mp + nq), \quad z = 2(mq - np), \quad d = m^2 + n^2 + p^2 + q^2. \quad (5)$$

First we remember that one of the two integers A and B satisfying the relation, $A^2 + B^2 = \square$, must be even.

Next, suppose y and z even, then (3) shows that c is even, and after removing the common factor 4 we find again that either $(y/2)^2$, or $(z/2)^2$ is a multiple of 4. But from (5) it is obvious that if x be odd three and only three of the numbers m, n, p, q must be odd or three even. This leads, however, to y and simultaneously z having the same factor 2, which, after suppressing in (3), leads to the conclusion again, that one number must be even, which is impossible.

Hence, the proposition is impossible.